

# A Criterion for Riemann Integrability with Bounded Derivatives

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## Abstract

In this note, we examine continuous functions on a closed interval that are differentiable on  $(a, b)$  and have bounded derivatives. More precisely, we show that such functions are uniformly continuous on  $[a, b]$  and hence Riemann integrable. We achieve this by showing uniform continuity by the Mean Value Theorem, and use this to control the difference between the largest and smallest values of the function on each subintervals. We conclude by discussing the limitations of the conditions considered and noting that they are stronger than necessary for Riemann integrability.

## 1 Introduction

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. An elementary way to approximate the area under the curve is by Riemann sums. This can be done by partitioning the interval into a finite number of subintervals and approximating the area over each subinterval with a rectangle. More precisely, the area of the rectangle on the  $i$ th subinterval is given by  $A_{\text{rec}}(i) = f(x_i^*)\Delta x_i$ , where  $x_i^*$  is an arbitrary point in  $[x_{i-1}, x_i]$  and  $\Delta x_i$  is the width of that subinterval. This method suggests that as the partition becomes refined, the corresponding Riemann sums approach the actual value of the area under the curve, which is represented by *definite integral*

$$\int_a^b f(x)dx.$$

However, not every bounded function necessarily yields a well-defined value when its partition is refined. This leads to the question, under what conditions does a bounded function produce a well-defined limit? In other words, when is a function Riemann integrable? The purpose of this note is to examine the conditions under which a bounded function is Riemann integrable. In particular, our focus is on functions that are differentiable on  $(a, b)$  with bounded derivative. This gives the following Theorem.

**Theorem.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ . If  $f$  is differentiable on  $(a, b)$  and  $f'$  is bounded on  $(a, b)$ , then  $f$  is Riemann integrable on  $[a, b]$ .*

## 2 Setup

We begin by establishing the necessary preliminaries, including definitions and notations that will be used throughout the remainder of this note.

**Definition 1.** Let  $a < b$  be real numbers. A partition  $P$  of  $[a, b]$  is a finite ordered set of points denoted by  $P = \{x_0, x_1, \dots, x_n\}$ , such that  $a = x_0 < x_1 < \dots < x_n = b$ .

**Definition 2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded, and let  $P$  be a partition of  $[a, b]$ . For each subinterval of  $[x_{i-1}, x_i]$ , let  $M_i$  be the supremum and  $m_i$  be the infimum of  $f$  on  $[x_{i-1}, x_i]$ . Then the upper sum and the lower sum are defined by

$$U(f, P) = \sum_{i=1}^n M_i \Delta x_i \quad L(f, P) = \sum_{i=1}^n m_i \Delta x_i$$

where  $\Delta x_i = x_i - x_{i-1}$ .

**Definition 3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. The function  $f$  is called Riemann integrable of  $[a, b]$  if

$$\inf\{U(f, P) : P \text{ is a partition of } [a, b]\} = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

Equivalently,  $f$  is Riemann integrable on  $[a, b]$  if for any  $\epsilon > 0$ , there exists a partition  $P$  of  $[a, b]$  such that

$$U(f, P) - L(f, P) < \epsilon.$$

We now recall the following classical result from calculus, which will be used to establish our main theorem.

**Theorem 1** (Mean Value Theorem). Let  $f$  be continuous over the closed interval  $[a, b]$  and differentiable over  $(a, b)$ . Then, there exists at least one point  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Proposition 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Suppose that  $f'$  is bounded on  $(a, b)$ . Then  $f$  is uniformly continuous on  $[a, b]$ .

*Proof.* Let  $x, y \in [a, b]$  such that  $x \neq y$ . Then, by Mean Value Theorem, there exists  $c \in (a, b)$  such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}.$$

Hence, by rearranging,  $f(x) - f(y) = f'(c)(x - y)$ , and by taking absolute values both sides gives  $|f(x) - f(y)| = |f'(c)||x - y|$ . Since  $f'$  is bounded, there exists  $M > 0$  such that  $|f'(x)| \leq M$  for every  $x \in (a, b)$ , and therefore  $|f(x) - f(y)| \leq M|x - y|$ . Now take arbitrary  $\epsilon > 0$ , and choose  $\delta = \epsilon/M$ . Then if  $|x - y| < \delta$ , we have  $|f(x) - f(y)| < \epsilon$ , which gives us that  $f$  is uniformly continuous on  $[a, b]$ .  $\square$

### 3 Main Result

The previous proposition shows that the bounded derivative of a function implies uniform continuity. Building on this, in this section we establish the main result of this note, which provides a sufficient condition for a function to be Riemann integrable on a closed interval.

**Theorem 2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$ . If  $f$  is differentiable on  $(a, b)$  and  $f'$  is bounded on  $(a, b)$ , then  $f$  is Riemann integrable on  $[a, b]$ .*

*Proof.* If  $a = b$ , then every partition has subintervals of width 0. Thus,  $\inf\{U(f, P)\} = \sup\{L(f, P)\} = 0$ , showing that  $f$  is Riemann integrable. Hence, we consider the case where  $a < b$ .

Let  $\epsilon > 0$  be arbitrary. Since  $f$  is differentiable on  $(a, b)$  and  $f'$  is bounded on  $(a, b)$ , by Proposition 1,  $f$  is uniformly continuous on  $[a, b]$ . By uniform continuity, for every  $\epsilon' > 0$ , there exists  $\delta > 0$  such that for all  $x, y \in [a, b]$ ,  $|x - y| < \delta$  implies  $|f(x) - f(y)| < \epsilon'$ . Now set  $\epsilon' = \frac{\epsilon}{b-a}$ . By substituting, we then have

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Now let  $P = \{x_0, x_1, \dots, x_n\}$  be a partition of  $[a, b]$  such that every subinterval has width less than  $\delta$ . That is,  $\Delta x_i < \delta$  for all  $i \in \{1, 2, \dots, n\}$ . We then choose any subinterval  $[x_{i-1}, x_i]$ . Then for any  $x, y \in [x_{i-1}, x_i]$ , we have  $|x - y| \leq \Delta x_i < \delta$ , and hence by uniform continuity,  $|f(x) - f(y)| < \frac{\epsilon}{b-a}$  for every pair of points  $x, y$  in the subinterval. This implies that the difference between the maximum and minimum possible value of  $f$  on  $[x_{i-1}, x_i]$  is less than  $\frac{\epsilon}{b-a}$ . In other words,  $|M_i - m_i| < \frac{\epsilon}{b-a}$  where  $M_i$  and  $m_i$  is supremum and infimum of  $f$  on  $[x_{i-1}, x_i]$ .

Now consider the difference between the upper and lower sums

$$U(f, P) - L(f, P) = \sum_{i=1}^n (M_i - m_i) \Delta x_i.$$

Using the inequality found above, we now have

$$U(f, P) - L(f, P) < \sum_{i=1}^n \left(\frac{\epsilon}{b-a}\right) (x_i - x_{i-1})$$

Note that the sum of the subintervals adds up to the total length,  $b - a$ . Thus,

$$\sum_{i=1}^n (x_i - x_{i-1}) = b - a.$$

Therefore,

$$U(f, P) - L(f, P) < \left(\frac{\epsilon}{b-a}\right) (b-a) = \epsilon.$$

Hence, by the definition of Riemann integrability, we conclude that  $f$  is Riemann integrable on  $[a, b]$ .  $\square$

In conclusion, the proof shows that a bounded derivative implies uniform continuity. This allows us to control how much the function varies on small intervals. When a function is restricted to sufficiently small subintervals, its difference between the maximum and minimum becomes small, which further allows that the difference between the upper and lower sums can be made arbitrarily small. As a result, the function is Riemann integrable on  $[a, b]$ .

However, this condition is not the strongest as possible. In fact, it is well known [1] that every continuous function on  $[a, b]$  is Riemann integrable, even if it is not differentiable. For example, the function  $f(x) = |x|$  on  $[-1, 1]$  is continuous but not differentiable at  $x = 0$ , yet it is Riemann integrable. Therefore, although the condition considered here is stronger than necessary, the result illustrates how bounded derivatives and uniform continuity lead to Riemann integrability.

## References

- [1] Michael Spivak, *Calculus*, 4th ed., Chapter 14, Publish or Perish, 2008.